

Bell's Theorem and Locality in Space

Igor V. Volovich*

*Department of Mathematics, Statistics and Computer Sciences
University of Vaxjo, S-35195, Sweden
igor.volovich@msi.vxu.se*

Abstract

Bell's theorem states that some quantum correlations can not be represented by classical correlations of separated random variables. It has been interpreted as incompatibility of the requirement of locality with quantum mechanics. We point out that in fact the space part of the wave function was neglected in the proof of Bell's theorem. However this space part is crucial for considerations of property of locality of quantum system. Actually the space part leads to an extra factor in quantum correlations and as a result the ordinary proof of Bell's theorem fails in this case. We present a criterium of locality in a realist theory of hidden variables. It is argued that predictions of quantum mechanics for Gaussian wave functions can be consistent with Bell's inequalities and hence Einstein's local realism is restored in this case.

*Permanent address: Steklov Mathematical Institute, Gubkin St.8, GSP-1, 117966, Moscow, Russia; volovich@mi.ras.ru

1 Introduction

Bell's theorem [1] states that there are quantum correlation functions that can not be represented as classical correlation functions of separated random variables. It has been interpreted as incompatibility of the requirement of locality with the statistical predictions of quantum mechanics [1]. For a recent discussion of Bell's theorem see, for example [2] - [18] and references therein. It is now widely accepted, as a result of Bell's theorem and related experiments, that "local realism" must be rejected.

Evidently, the very formulation of the problem of locality in quantum mechanics is based on ascribing a special role to the position in ordinary three-dimensional space. It is rather surprising therefore that the space dependence of the wave function is neglected in discussions of the problem of locality in relation to Bell's inequalities. Actually it is the space part of the wave function which is relevant to the consideration of the problem of locality.

In this note we point out that the space part of the wave function leads to an extra factor in quantum correlation and as a result the ordinary proof of Bell's theorem fails in this case. We present a criterium of locality (or nonlocality) of quantum theory in a realist model of hidden variables. We argue that predictions of quantum mechanics can be consistent with Bell's inequalities for Gaussian wave functions and hence Einstein's local realism is restored in this case.

2 Bell's Theorem

Consider a pair of spin one-half particles formed in the singlet spin state and moving freely in opposite directions. If one neglects the space part of the wave function then the quantum mechanical correlation of two spins in the singlet state ψ_{spin} is

$$E_{spin}(a, b) = \langle \psi_{spin} | \sigma \cdot a \otimes \sigma \cdot b | \psi_{spin} \rangle = -a \cdot b \quad (1)$$

Here a and b are two unit vectors in three-dimensional space and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Bell's theorem states that the function $E_{spin}(a, b)$ (1) can not be represented in the form

$$P(a, b) = \int A(a, \lambda) B(b, \lambda) d\rho(\lambda) \quad (2)$$

Here $A(a, \lambda)$ and $B(b, \lambda)$ are random fields on the sphere, $|A(a, \lambda)| \leq 1$, $|B(b, \lambda)| \leq 1$ and $d\rho(\lambda)$ is a positive probability measure, $\int d\rho(\lambda) = 1$. The parameters λ are interpreted as hidden variables in a realist theory.

One has the following Bell-Clauser-Horn-Shimony-Holt (CHSH) inequality

$$|P(a, b) - P(a, b') + P(a', b) + P(a', b')| \leq 2 \quad (3)$$

From the other hand there are such vectors ($ab = a'b = a'b' = -ab' = \sqrt{2}/2$) for which one has

$$|E_{spin}(a, b) - E_{spin}(a, b') + E_{spin}(a', b) + E_{spin}(a', b')| = 2\sqrt{2} \quad (4)$$

Therefore if one supposes that $E_{spin}(a, b) = P(a, b)$ then one gets the contradiction.

3 Criterium of Locality

In the previous section the space part of the wave function of the particles was neglected. However exactly the space part is relevant to the discussion of locality. The complete wave function is $\psi = (\psi_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2))$ where α and β are spinor indices and \mathbf{r}_1 and \mathbf{r}_2 are vectors in three-dimensional space.

We suppose that detectors are located within the two localized regions \mathcal{O}_1 and \mathcal{O}_2 respectively, well separated from one another. Quantum correlation describing the measurements of spins at the localized detectors is

$$E(a, \mathcal{O}_1, b, \mathcal{O}_2) = \langle \psi | \sigma \cdot a P_{\mathcal{O}_1} \otimes \sigma \cdot b P_{\mathcal{O}_2} | \psi \rangle \quad (5)$$

Here $P_{\mathcal{O}}$ is the projection onto the region \mathcal{O} . Let us consider the case when the wave function has the form $\psi = \psi_{spin} \phi(\mathbf{r}_1, \mathbf{r}_2)$. One has

$$E(a, \mathcal{O}_1, b, \mathcal{O}_2) = g(\mathcal{O}_1, \mathcal{O}_2) E_{spin}(a, b) \quad (6)$$

where the function

$$g(\mathcal{O}_1, \mathcal{O}_2) = \int_{\mathcal{O}_1 \times \mathcal{O}_2} |\phi(\mathbf{r}_1, \mathbf{r}_2)|^2 d\mathbf{r}_1 d\mathbf{r}_2 \quad (7)$$

describes correlation of particles in space. Note that one has

$$0 \leq g(\mathcal{O}_1, \mathcal{O}_2) \leq 1 \quad (8)$$

To investigate the property of locality in a realist theory of hidden variables we will study whether the quantum correlation (6) can be represented in the form (2). More exactly one inquires whether one can write the representation

$$g(\mathcal{O}_1, \mathcal{O}_2)E_{spin}(a, b) = \int A(a, \mathcal{O}_1, \lambda)B(b, \mathcal{O}_2, \lambda)d\rho(\lambda) \quad (9)$$

We have indicated a possible dependence of A on the region \mathcal{O}_1 and B on \mathcal{O}_2 . A possible dependence of the measure $d\rho(\lambda)$ on the location of detectors \mathcal{O}_1 and \mathcal{O}_2 deserves a further discussion.

Note that if we set $g(\mathcal{O}_1, \mathcal{O}_2) = 1$ in (9) as it was done in the original proof of Bell's theorem, then it means we did a special preparation of the states of particles to be completely localized inside of detectors. It seems the entanglement of the original states can be destroyed in the process of such a preparation.

Now from (3), (4) and (9) one obtains the following *criterium of locality* in a realist theory of hidden variables

$$g(\mathcal{O}_1, \mathcal{O}_2) \leq 1/\sqrt{2} \quad (10)$$

If the inequality (10) is valid for regions \mathcal{O}_1 and \mathcal{O}_2 which are well separated from one another then there is no violation of the CHSH inequalities (3) and therefore there is no violation of locality in the corresponding state ψ . From the other side, if for a pair of well separated regions \mathcal{O}_1 and \mathcal{O}_2 one has

$$g(\mathcal{O}_1, \mathcal{O}_2) > 1/\sqrt{2} \quad (11)$$

then it could be violation of the realist locality in these regions for a given state.

4 Gaussian Wave Functions

Now let us consider the criterium of locality for Gaussian wave functions. We will show that with a reasonable accuracy there is no violation of locality in this case. Let us take the wave function ϕ of the form $\phi = \psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)$ where the individual wave functions have the moduli

$$|\psi_1(\mathbf{r})|^2 = \left(\frac{m^2}{2\pi}\right)^{3/2} e^{-m^2 \mathbf{r}^2/2}, |\psi_2(\mathbf{r})|^2 = \left(\frac{m^2}{2\pi}\right)^{3/2} e^{-m^2 (\mathbf{r}-1)^2/2} \quad (12)$$

We suppose that the length of the vector \mathbf{l} is much larger than $1/m$. We can make measurements of $P_{\mathcal{O}_1}$ and $P_{\mathcal{O}_2}$ for any well separated regions \mathcal{O}_1 and \mathcal{O}_2 . Let us suppose a rather nonfavorite case for the criterium of locality when the wave functions of the particles are almost localized inside the regions \mathcal{O}_1 and \mathcal{O}_2 respectively. In such a case the function $g(\mathcal{O}_1, \mathcal{O}_2)$ can take values near its maximum. We suppose that the region \mathcal{O}_1 is given by $|r_i| < 1/m$, $\mathbf{r} = (r_1, r_2, r_3)$ and the region \mathcal{O}_2 is obtained from \mathcal{O}_1 by translation on \mathbf{l} . Hence $\psi_1(\mathbf{r}_1)$ is a Gaussian function with modules appreciably different from zero only in \mathcal{O}_1 and similarly $\psi_2(\mathbf{r}_2)$ is localized in the region \mathcal{O}_2 . Then we have

$$g(\mathcal{O}_1, \mathcal{O}_2) = \left(\frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-x^2/2} dx \right)^6 \quad (13)$$

One can estimate (13) as

$$g(\mathcal{O}_1, \mathcal{O}_2) < \left(\frac{2}{\pi} \right)^3 \quad (14)$$

which is smaller than $1/\sqrt{2}$. Therefore the locality criterium (10) is satisfied in this case.

5 Conclusions

It is shown in this note that if we do not neglect the space part of the wave function of two particles then the prediction of quantum mechanics for the correlation with the Gaussian space parts of the wave functions can be consistent with Bell's inequalities. One can say that Einstein's local realism is restored in this case.

We did only a rough estimate and it would be interesting to investigate whether one can prepare a reasonable wave function for which the condition of nonlocality (11) is satisfied for a pair of the well separated regions. In principle the function $g(\mathcal{O}_1, \mathcal{O}_2)$ can approach its maximal value 1 if the wave functions of the particles are very well localized within the detector regions \mathcal{O}_1 and \mathcal{O}_2 respectively. However, perhaps to establish such a localization one has to destroy the original entanglement because it was created far away from detectors. If D_1 is a region which contains the place where the entangled state was created and the path to the detector \mathcal{O}_1 then the conditional probability of finding the particle 1 in the region \mathcal{O}_1 with the projection of spin along

vector a is given by Bayes's formula $P(a, \mathcal{O}_1|D_1) = P(a, \mathcal{O}_1, D_1)/P(D_1)$. It is especially important if one could prepare such strictly localized within \mathcal{O}_1 and \mathcal{O}_2 entangled states in real experiments. Then one could say that "local realism" must be rejected.

One has to study the dependence of the wave function not only on the space variables but also on time. It seems the function $g(\mathcal{O}_1, \mathcal{O}_2)$ and its relativistic analogue should be taken into account in discussions of experiments performed in the studying of quantum nonlocality. In particular it gives a contribution to the efficiency of detectors.

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